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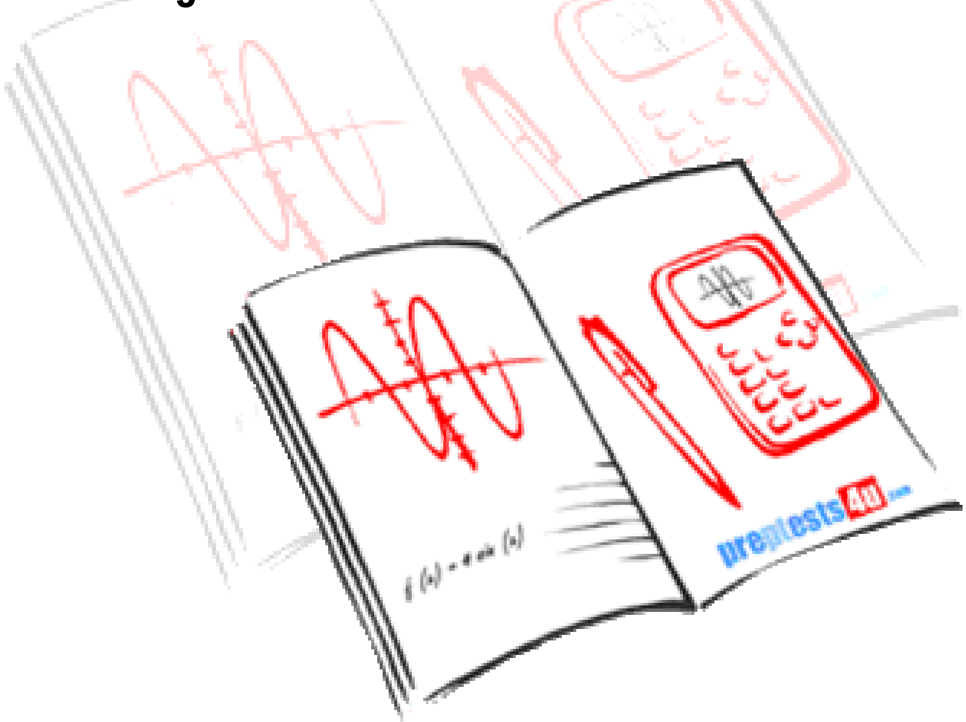
Calculus 1

Chapter 4 - The Concept of Integration

Area and Average Value of a Function



Area and Average Value of a Function

The following formula evaluates the area between two curves $f(x)$ and $g(x)$ bounded by lines $x = a$ and $x = b$.

$$A = \int_a^b |f(x) - g(x)|dx$$

The absolute value takes care of both conditions $f(x) \geq g(x)$ or $g(x) \geq f(x)$ on $[a,b]$. If one of the functions $f(x)$ or $g(x)$ is equal to zero, then

$A = \int_a^b |f(x)|dx$ or $A = \int_a^b |g(x)|dx$ will be the area under a curve on $[a,b]$.

Example 1:

Find the area under the curve $f(x) = 3x - 2$, $x = -2$ and $x = 4$.

Solution:

$$A = \int_{-2}^{4} (3x - 2)dx = \left[\frac{3x^2}{2} - 2x\right]_{-2}^{4} = 6$$

Example 2:

Find the area under the curve $f(x) = |3x - 2|$, $x = -2$ and $x = 4$.

Solution:

Since the absolute value is given here, we must find out the signs of $f(x)$ on the given interval.

$$3x - 2 = 0, x = \frac{2}{3}$$

$f(x) \prec 0$ on $\left(-2, \frac{2}{3}\right)$ and $f(x) \geq 0$ on $\left[\frac{2}{3}, 4\right]$, hence

$$A = \int_{-2}^{4} f(x)dx = -\int_{-2}^{\frac{2}{3}} (3x-2)dx + \int_{\frac{2}{3}}^{4} (3x-2)dx$$

To evaluate these integrals, follow the solutions in Example 1.

Example 3:

Find the area under the curve $f(x) = x^2 - 3x + 2$, $x = -1$ and $x = 2$.

Solution:

$$A = \int_{-1}^{2} (x^2 - 3x + 2)dx = \left[\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right]_{-1}^{2} = \frac{9}{2}$$

Example 4:

Find the area under the curve $f(x) = \left|x^2 - 3x + 2\right|$, $x = -1$ and $x = 3$.

Solution:

Since the absolute value is given here, we must find out the sign of $f(x)$ on the given interval.

$$x^2 - 3x + 2 = (x-1)(x-2) = 0, x = 1, x = 2$$

$f(x) \prec 0$ on $(1,2)$ and $f(x) \geq 0$ on $[-1,1]U[2,3]$

Hence,

$$A = \int_{-1}^{3} f(x)dx = \int_{-1}^{+1} (x^2 - 3x + 2)dx - \int_{1}^{2} (x^2 - 3x + 2)dx + \int_{2}^{3} (x^2 - 3x + 2)dx$$

To evaluate these integrals, follow the solutions in Example 1.

Example 5:

Find the area bounded by $f(x)=\dfrac{e^x+1}{x+e^x}, x=0, x=Ln3.$

Solution:

$$A=\int_0^{Ln3} f(x)dx=\int_0^{Ln3}\frac{e^x+1}{x+e^x}dx$$

Let

$$u=x+e^x,\ du=(1+e^x)dx,\ u(0)=1, u(Ln3)=3+Ln3$$

Hence,

$$A=\int_0^{Ln3} f(x)dx=\int_0^{Ln3}\frac{e^x+1}{x+e^x}dx=\int_1^{3+Ln3}\frac{du}{u}=[Lnu]_1^{3+Ln3}=Ln(3+Ln3)\cong 1.4106$$

Exercise:

Find the area bounded by $f(x)=\dfrac{e^{-x}-1}{x+e^{-x}}, x=0, x=Ln3.$

Average Value of a Function

Average value of a function is defined as:

$$f_{ave}=\frac{1}{b-a}\int_a^b f(x)dx$$

Example 6:

Find the average value of $f(x)$ on $[-2,3]$

$$f(x)=\begin{cases} e^x+3, x\geq 0 \\ x^2+3x+4, x\prec 0\end{cases}$$

Solution:

$$f_{ave}=\frac{1}{2}\int_{-2}^{0}(x^2+3x+4)dx+\frac{1}{3}\int_0^3(e^x+3)dx$$

Evaluations of these integrals are very straight forward. However, it is absolutely important to identify the signs of $f(x)$ on the given interval before any attempt to do the integration.