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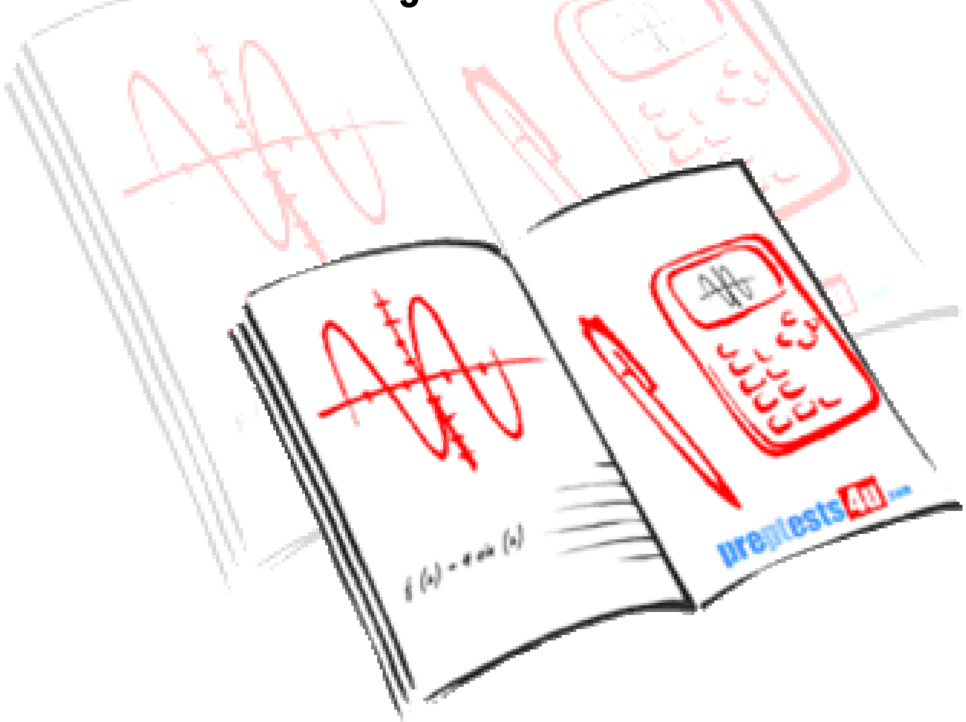
Calculus 1

Chapter 3 - The Concept of Differentiation

Derivative of Functions Using Formulas



Derivative of Functions Using Formulas

Derivative formulas, given in the Appendix, are applied to different equations in order to demonstrate their crucial rule in finding derivatives of more complex functions.

Example 1:

Find the derivative of $f(x) = 3x^4 - 4x^2 - 1$ at $x = 0,1$

Solution:

Using appropriate formula,

$$f'(x) = 12x^3 - 8x$$

$$f'(0) = 0$$

$$f'(1) = 12 - 8 = 4$$

The values of $f'(0) = 0$ and $f'(1) = 4$ are the slopes to the function $f(x)$ at points $(0,-1)$ and $(1,-2)$, respectively.

Example 2:

Given $f(x) = \dfrac{3x^2 - 4x + 2}{1 - 2x}$, find $f'(0) = ?$

Solution:

Using the formula of derivative for the fractional equations, we reach the following:

$$f'(x) = \frac{(3x^2 - 4x + 2)'(1 - 2x) - (3x^2 - 4x + 2)(1 - 2x)'}{(1 - 2x)^2}$$

$$f'(x) = \frac{(6x - 4)\ (1 - 2x) - (3x^2 - 4x + 2)(-2x)}{(1 - 2x)^2}$$

Warning:
You must complete the process of finding derivative before substituting any value for x.

$$f'(0) = \frac{-4+8}{(1)^2} = 4$$

Note:
The slope of a line tangent to $f(x)$ at $(0,2)$ is $m = 4$; therefore, the equation of the tangent line is $y = 4x + 2$.

Example 3:

Find the point(s) of x where function $f(x) = 3x^2\sqrt{x+1}$ has horizontal tangent lines.

Solution:

$$f'(x) = 0$$

$$f(x) = 3x^2\sqrt{x+1} = \sqrt{9x^5 + 9x^4} = \left(9x^5 + 9x^4\right)^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}\left(45x^4 + 36x^3\right)\left(9x^5 + 9x^4\right)^{\frac{-1}{2}} = \frac{45x^4 + 36x^3}{2\sqrt{9x^5 + 9x^4}}$$

$$f'(x) = \frac{1}{2}\left(45x^4 + 36x^3\right)\left(9x^5 + 9x^4\right)^{\frac{-1}{2}} = \frac{15x^2 + 12x}{2\sqrt{x+1}} = 0$$

Hence, $15x^2 + 12x = 0$**,** $x = 0, -\frac{4}{5}$**, both values are in the domain of** $f(x)$**.**

Note:
You should always check the values found for validity.

Example 4:

Find the equation of the tangent line to $f(x)=\dfrac{e^{x^2}}{1-e^{x^2}}$ at $x=0$.

Solution:

$$f'(x)=\frac{\left(e^{x^2}\right)'\left(1-e^{x^2}\right)-\left(1-e^{x^2}\right)'\left(e^{x^2}\right)}{\left(1-e^{x^2}\right)^2}$$

$$f'(x)=\frac{\left(2xe^{x^2}\right)\left(1-e^{x^2}\right)-\left(-2xe^{x^2}\right)\left(e^{x^2}\right)}{\left(1-e^{x^2}\right)^2}=\frac{2xe^{x^2}}{\left(1-e^{x^2}\right)^2}$$

$$f'(0)=\frac{0}{0}$$

To resolve this situation, we need to take the limit of $f'(x)$ as $x\to 0$

$$\lim f'(x)=\lim\frac{2e^{x^2}+4x^2e^{x^2}}{2\left(-2xe^{x^2}\right)\left(1-e^{x^2}\right)}=undefined$$

$$x\to 0$$

Note:

Line $x=0$ is a vertical asymptote. Function $f(x)$ is not defined at $x=0$.

Example 5:

Find the equation of tangent line(s) to $f(x)=\dfrac{x^2-3x+1}{2x}$ and perpendicular to the line $x=5$.

Solution:

Slope of the vertical line $x=5$ is $m_1=\infty=undefined$, hence, the slope of the tangent line to $f(x)$ must be $m_2=0$.Condition of perpendicular is $m_1 * m_2=-1$.

Therefore,

$$f'(x) = \frac{(x^2 - 3x + 1)'(2x) - (x^2 - 3x + 1)(2x)'}{(2x)^2} = \frac{x^2 - 1}{2x^2} = 0$$

Hence,

$$x^2 - 1 = 0, \quad x = \pm 1.$$

Points $\left(1, -\frac{1}{2}\right)$ and $\left(-1, -\frac{5}{2}\right)$ are the only two points of $f(x)$ where tangent lines are perpendicular to the given line.

Equations of tangent lines are:

$$y_1 = f(1) = -\frac{1}{2} \quad \text{and} \quad y_2 = f(-1) = -\frac{5}{2}$$

Note:
Since $x = 5$ is a vertical line, any line perpendicular to this line must be a horizontal line. Lines y_1 and y_2 are horizontal lines.

Exercise: Do this problem replacing the word perpendicular with parallel.

L'Hospitals' Rule

Suppose $f(x)$ and $g(x)$ are differentiable at the vicinity of a and $g'(x) \neq 0$.

If

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \text{ or } \frac{\infty}{\infty}$$

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

The following examples will further clarify this important rule.

Example 6:

Apply the L'Hospitals' Rule to find the limit of the following function.

$$f(x) = \frac{e^{x^2} - 1}{x^2}$$

Solution:

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{e^{x^2} - 1}{x^2} = \frac{0}{0}$$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\left(e^{x^2} - 1\right)'}{\left(x^2\right)'} = \lim_{x \to 0} \frac{2xe^{x^2}}{2x} = \lim_{x \to 0} e^{x^2} = 1$$

Example 7:

Apply the L'Hospitals' Rule to find the limit of the following function.

$$f(x) = \frac{Cos^2 x - 1}{x^2}$$

Solution:

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{Cos^2 x - 1}{x^2} = \frac{0}{0}$$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\left(Cos^2 x - 1\right)'}{\left(x^2\right)'} = \lim_{x \to 0} \frac{-2Cosx * Sinx}{2x} = -\lim_{x \to 0} \frac{Sinx}{x} * \lim_{x \to 0} Cosx = -1$$

Example 8:

Find $$\lim_{x \to 0}\left(\frac{1}{x^2}-\frac{1}{x^4}\right)=\infty-\infty$$

Solution:

$$\lim_{x \to 0}\left(\frac{x^2-1}{x^4}\right)=\lim_{x \to 0}\left(\frac{1}{x^2}\right)-\lim_{x \to 0}\left(\frac{1}{x^4}\right)=0-\infty=-\infty$$

Exercise: $$\lim_{x \to 1}\frac{x^e-1}{x^2-1}=?$$

Definition

Function $f(x)$ is differentiable at $x=a$ if $f'(a)$ exists.

Note:
$f(x)$ is differentiable at $x=a$, then $f(x)$ is continuous at $x=a$. The reverse may not be true. The following examples clarify this matter.

Example 9:

Given $f(x) = \begin{cases} e^x + 3, x \geq 0 \\ x^2 + 3x + 4, x \prec 0 \end{cases}$

a. Determine if $f(x)$ is **continuous at** $x = 0$

Solution:

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (e^x + 3) = 4$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x^2 + 3x + 4) = 4$$

$$f(0) = 4$$

Conclusion: **Yes, $f(x)$ is continuous at $x = 0$.**

b. Determine if $f(x)$ is **differentiable** at $x = 0$

Solution:
We need to find $f'(0) = ?$

$$f'_+(0) = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{(e^h + 3) - 4}{h} = \lim_{h \to 0^+} \frac{e^h - 1}{h} = 1$$

$$f'_-(0) = \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{(h^2 + 3h + 4) - 4}{h} = \lim_{h \to 0^-} (h + 3) = 3$$

Conclusion: $f'(0)$ **does not exists.** $f'(x) = \begin{cases} (e^x), x \succ 0 \\ (2x + 3), x \prec 0 \end{cases}$

$f(x)$ **is not differentiable at** $x = 0$

Example 10:

Given $f(x) = \begin{cases} 1 - x^2, x \prec 0 \\ 1 - Sin2x, x \geq 0 \end{cases}$

a. Determine if $f(x)$ is **continuous at** $x = 0$

Solution:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (1 - x^2) = 1$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (1 - Sin2x) = 1$$

$$f(0) = 1$$

Conclusion: **Yes,** $f(x)$ **is continuous at** $x = 0$

b. Determine if $f(x)$ is **differentiable** at $x = 0$.

Solution:
We need to find $f'(0) = ?$

$$f'_-(0) = \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{(1-h^2) - 1}{h} = \lim_{h \to 0^-} (-h) = 0$$

$$f'_+(0) = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{(1 - Sin2h) - 1}{h} = \lim_{h \to 0^+} \frac{-2Sinh * Cosh}{h} = -2$$

Conclusion: $f(x)$ **is not differentiable at** $x = 0$ **since** $f'_-(0) \neq f'_+(0)$ **.**

$$f'(x) = \begin{cases} (-2x), x \prec 0 \\ (-2Cos2x), x \succ 0 \end{cases}$$

Note:
To find out if a function is differentiable at a point, take derivative of the function and evaluate the limits of the derivative at the point given from left and right. If these derivatives exist and equal to each other, then given function is differentiable at that point. This condition automatically constitutes the continuity at that point. The reverse may not be true.

Example 11:

Given $f(x) = \begin{cases} 1 + x - 2x^2, x \geq 0 \\ x + 1, x \prec 0 \end{cases}$

a. Determine if $f(x)$ is **continuous at** $x = 0$

Solution:

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+}(1 + x - 2x^2) = 1$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-}(x + 1) = 1$$

$$f(0) = 1$$

Conclusion: **Yes,** $f(x)$ **is continuous at** $x = 0$.

b. Determine if $f(x)$ is **differentiable** at $x=0$.

Solution:
We need to find $f'(0)=?$

$$f'_{+}(0)=\lim_{h\to 0^{+}}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^{+}}\frac{(1+h-2h^{2})-1}{h}=\lim_{h\to 0^{+}}\frac{h(1-2h)}{h}=1$$

$$f'_{-}(0)=\lim_{h\to 0^{-}}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^{-}}\frac{(h+1)-1}{h}=1$$

Conclusion: $f(x)$ **is differentiable at** $x=0$ **since** $f'_{+}(0)=f'_{-}(0)=1$.

Note:
Since $f(x)$ **is differentiable at** $x=0$**, it is also continuous at** $x=0$**.**
Reverse is not necessarily true as indicated in the previous examples.

$$f'(x)=\begin{cases}(1-4x), x\geq 0\\ 1, x\prec 0\end{cases}$$