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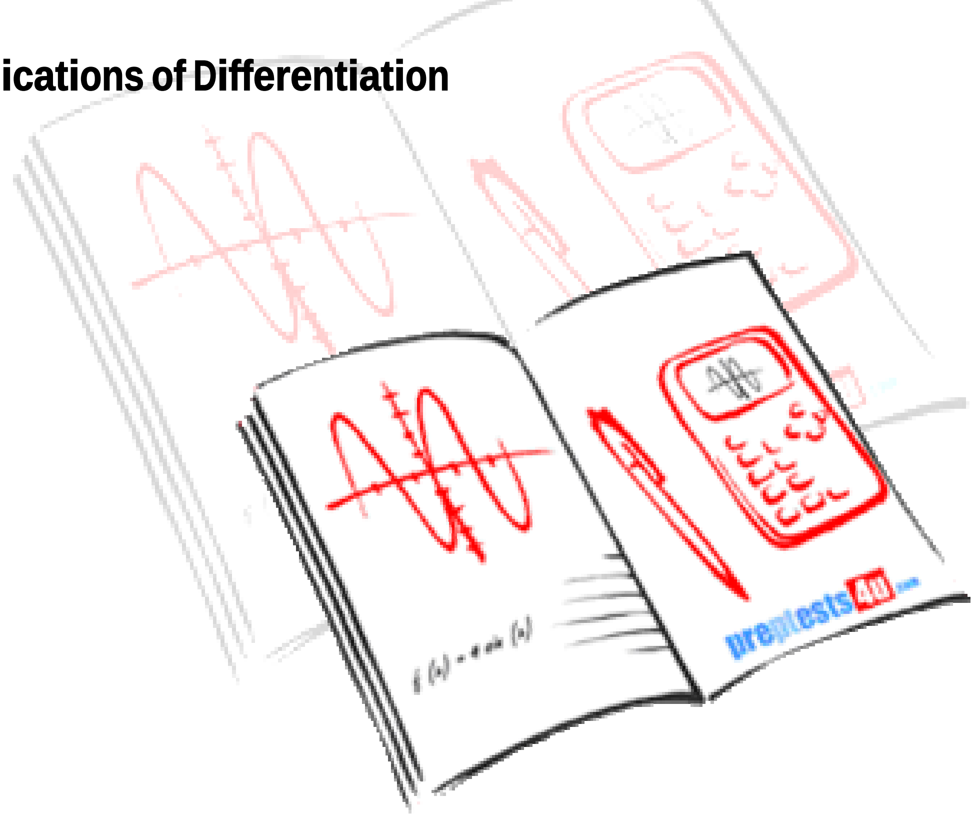
Calculus 1

Chapter 3 - The Concept of Differentiation

Applications of Differentiation



Applications of Differentiation

Applications of differentiation to the following topics are given throughout the examples.

- a. **Critical Points**
- b. **Local Maximum and Minimum**
- c. **Absolute Maximum and Minimum**
- d. **Intervals of Increasing and Decreasing**
- e. **Inflection Points**
- f. **Test for Concavity**
- g. **The Mean Value Theorem**

Assumptions:

For a-g, it is assumed that the function $f(x)$ is continuous on a closed interval $[a,b]$. For g, in addition to continuity, it is also assumed that function $f(x)$ is differentiable on the open interval (a,b).

a. Critical points:

To find the critical point(s) of $f(x)$, let $f'(x)=0$ or $f'(x)=\pm\infty$.

b. Absolute Maximum and Minimum / Local Maximum and Minimum:

To find absolute Maximum or Minimum, evaluate $f(x)$ at a, b, and all critical points. The highest value is absolute maximum and the lowest value is absolute minimum. The local maximum or/and minimum are determined by evaluating $f(x)$ at the critical points.

If sign of $f'(x)$ changes from positive to negative on a given point c $\in (a,b)$, then $f(x)$ has a local maximum at c.

If sign of $f'(x)$ changes from negative to positive on a given point c $\in (a,b)$, then $f(x)$ has a local minimum at c.

c. Intervals of Increasing and Decreasing:

If $f'(x)\succ 0$ on a given interval, then $f(x)$ is increasing in that interval.
If $f'(x)\prec 0$ on a given interval, then $f(x)$ is decreasing in that interval.

d. Test for Concavity:

If $f''(x) \succ 0$ for all x belong to a given interval, then the graph of $f(x)$ is concave up or upward on this interval.

If $f''(x) \prec 0$ for all x belong to a given interval, then the graph of $f(x)$ is concave down or downward on this interval.

e. Second derivative tests for local minimum and local maximum:

If c is a critical point, $f'(c) = 0$, and $f''(x) \succ 0$, then $f(x)$ has a local maximum at $x = c$.

If c is a critical point, $f'(c) = 0$, and $f''(x) \prec 0$, then $f(x)$ has a local minimum at $x = c$.

f. The Mean Value Theorem:

If $f(x)$ is continuous on $[a,b]$ and $f(x)$ is differentiable on (a,b), then There is a point $c \in (a,b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Example 1:

Find the local maximum and minimum for the function $f(x) = 2x^3 - 5x^2 - 4x + 1$.

Solution:

$$f'(x) = 6x^2 - 10x - 4 = 2(3x+1) * (x-2) = 0$$

$x = 2, -\frac{1}{3}$, critical points

Second derivative test for local minimum and maximum,

$$f''(x) = 12x - 10$$

$f''(2) = 14 \succ 0$, hence local minimum at $x = 2$

$f''\left(-\frac{1}{3}\right) = -14 \prec 0$, hence local maximum at $x = -\frac{1}{3}$

Example 2:

Find the absolute maximum and minimum for the function $f(x)$ given in example 1 on the interval $[0,3]$.

Solution:

Critical points found in example 1, $x = 2, -\frac{1}{3}$

Since $x = -\frac{1}{3} \notin [0,3]$**, it is discarded.**

$f(0) = -1$, **Absolute Maximum**

$f(2) = -13$, **Absolute Minimum**

$f(3) = -4$

Example 3:

Find the value(s) of x at which function $f(x) = xe^{-x^2}$ has **relative extrema,** (local Max./Min.).

Solution:

$$f'(x) = e^{-x^2} - 2x^2 e^{x^2} = e^{-x^2}\left(1 - 2x^2\right) = 0, \quad e^{-x^2} \neq 0$$

$x = \pm\frac{\sqrt{2}}{2}$, critical points

Second derivative test for local minimum and maximum,

$$f''(x) = -4xe^{-x^2} + (-2x)e^{-x^2}\left(1 - 2x^2\right) = -2x\left(3 - 2x^2\right)e^{-x^2}$$

$f''\left(\frac{\sqrt{2}}{2}\right) \prec 0$, Local Maximum at $x = \frac{\sqrt{2}}{2}$

$f''\left(-\frac{\sqrt{2}}{2}\right) \succ 0$, Local Minimum at $x = -\frac{\sqrt{2}}{2}$

Example 4:

Find the absolute maximum and minimum for the function $f(x)$, in example 3, on the interval $[-1,0]$.

Solution:

Note:

Function $f(x)$ is continuous on $[-1,0]$. Discard $x = \frac{\sqrt{2}}{2}$.

$$f\left(-\frac{\sqrt{2}}{2}\right) = -\frac{1}{\sqrt{2e}}$$, **Absolute Minimum**

And

$f(0) = 0$, **Absolute Maximum**

$f(-1) = -\frac{1}{e}$,

Example 5:

Find the maximum and minimum for the function $f(x) = \cos^2 x - 3\sin x$ on the interval $[0,2\pi]$.

Solution:

Notice that $f(x)$ is continuous on $[0,2\pi]$.

$$f'(x) = -2\cos x \sin x - 3\cos x = (\cos x)(-2\sin x - 3) = 0$$

$\cos x = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$ critical points.

$-2\sin x - 3 = 0 \Rightarrow \sin x = -\frac{3}{2} \prec -1$, **hence no solution**.

Remember:

$$-1 \le \sin x \le 1$$

$f(0) = 1$, **Absolute Maximum**

$f\left(\frac{\pi}{2}\right) = -3$, **Absolute Minimum**

$f(2\pi) = 1$, **Absolute Maximum**

Example 6: **(The Mean Value Theorem)**

Apply the Mean Value Theorem to the following function on the given interval.

$$f(x) = x + \frac{1}{x^2} \quad , \quad [2,5]$$

Solution:

First check if the conditions for using The Mean Value Theorem apply:

a. $f(x)$ **is continuous on** $[2,5]$
b. $f(x)$ **is differentiable on** $(2,5)$

Both conditions are satisfied, hence

$$f(5) = 5 + \frac{1}{25} = 5.04 \quad \text{and} \quad f(2) = 2 + \frac{1}{4} = 2.25$$

$$f'(x) = 1 - \frac{2}{x^3} = \frac{f(5) - f(2)}{5 - 2} = \frac{5.04 - 2.25}{3} \cong 0.93$$

$$1 - \frac{2}{x^3} \cong 0.93 \Rightarrow x \cong 3.051,$$

Note:
$3.051 \in (2,5)$

Example 7: **(The Mean Value Theorem)**

Check if The Mean Value Theorem is applicable to the following function on the given interval.

$$f(x) = x + \frac{1}{x^2} \quad , \quad [-1,2]$$

Solution:

First check if the conditions for using The Mean Value Theorem apply:

$f(x)$ is not continuous at $x = 0 \in [-1,2]$, hence
The Mean Value Theorem does not apply.

Note:
If a function is continuous at a point, then it is not necessarily differentiable at that point. But if a function is differentiable at a point, then the function is certainly continuous at that point. See the next example.

Example 8:

Check if The Mean Value Theorem is applicable to the following function on the given interval.

$$f(x) = \begin{cases} e^x + 3, & x \geq 0 \\ x^2 + 3x + 4, & x < 0 \end{cases} \quad , \quad [-1,+1]$$

Solution:

First check if the conditions for using The Mean Value Theorem apply:

a. Function $f(x)$ is continuous on $[-1,+1]$
b. Function $f(x)$ is not differentiable at $x = 0 \in (-1,+1)$.

Conclusion: The Mean Value Theorem does not apply here due to part b.

See Example 9, Section 2, Chapter 2 for the details of the differentiability.

Example 9:

Find the intervals, at which function $f(x) = x^3 - 3x^2 - 8x$ is increasing, decreasing, concave up and concave down.

Solution:

$$f'(x) = 3x^2 - 6x - 24 = 3(x^2 - 2x - 8) = 3(x-4)(x+2) = 0$$

$x = -2, 4$, **Critical Points**

$f''(x) = 6x - 6 = 6(x-1) = 0, \Rightarrow x = 1$, **Inflection Point**

Interval of Decreasing, $f'(x) \prec 0$, $(-2,4)$
Interval of Increasing, $f'(x) \succ 0$, $(-\infty,-2) U (4,+\infty)$
Interval of Concave up, $f''(x) \succ 0$, $(1,\infty)$
Interval of Concave down, $f''(x) \prec 0$, $(-\infty,1)$

Note:
From the given intervals, one may easily answer the following questions:

a. Interval at which $f(x)$ is increasing and concave up: $(4,\infty)$
b. Interval at which $f(x)$ is decreasing and concave down: $(-2,1)$

Exercise:

c. Interval at which $f(x)$ is increasing and concave down:______________
d. Interval at which $f(x)$ is decreasing and concave up: _______________

Example 10:

A rectangular piece of land has dimensions x by y. We want to build a symmetric rectangular tennis court inside the land which has a margin a from x and margin b from y on both sides. Assuming the area of the land is given as C square feet, find:

a. Constraint

Solution:

$$x * y = C$$

b. Maximum area of the tennis court

Solution:

$$A_t = (x-2a)*(y-2b), \text{ also } \quad y = \frac{C}{x}$$

$$A_t = (x-2a)*\left(\frac{C}{x}-2b\right) = C-2bx-\frac{2aC}{x}+4ab$$

$$A_t^{'} = -2b+\frac{2aC}{x^2} = 0 \Rightarrow -2bx^2+2aC = 0 \Rightarrow x = \sqrt{\frac{aC}{b}}, y = \sqrt{\frac{bC}{a}}$$

Example 11:

In a right triangle with hypotenuse y, adjacent side x and fixed height h, find the rate of change of y with respect to the rate of change of x.

Solution:

$$y^2 = x^2 + h^2, \; x = x(t) \text{ and } y = y(t)$$

Using chain rule:

$$2\frac{dy}{dx}*\frac{dx}{dt}*y = 2x\frac{dx}{dt}+0$$

$$\frac{dy}{dt} = \frac{x}{y}*\frac{dx}{dt}$$

Cost Function

1. Cost function is denoted by $C(x)$.
2. Average cost function $= c(x) = \frac{C(x)}{x}$
3. Marginal cost function, $c^{'}(x)$
4. If average cost is minimum, then Marginal cost = Minimum of average cost
5. Demand function, price function, $q(x)$, where x is number of units produced.
6. Revenue function $= R(x) = x*q(x)$
7. Marginal Revenue function $= R^{'}(x)$

8. Profit function $= P(x) = R(x) - C(x)$
9. Marginal profit function $= P'(x) = R'(x) - C'(x)$
10. Maximize profit $\Rightarrow P'(x) = 0 \Rightarrow C'(x) = R'(x)$
11. Supply function, $P_{\sup ply}(x)$
12. Producer surplus at equilibrium point: $\int_0^x [P - P_s(x)]dx$ where P is the price at the equilibrium point, calculated by finding the equilibrium point and substituting this into the demand function. To find the equilibrium point, let demand function equals the supply function.

Example 12:

The demand function for a product is given as $q(x) = 100 - \frac{x}{1000}$ and the cost function is $C(x) = 0.04x^2 + x + 200$. Find:

a. Average Cost Function

Solution:

$$c(x) = \frac{C(x)}{x} = \frac{0.04x^2 + x + 200}{x} = 0.04x + 1 + \frac{200}{x}$$

b. Marginal Cost Function

Solution:

$$C'(x) = 0.08x + 1$$

c. Revenue fFunction

Solution:

$$R(x) = x * q(x) = 100x - \frac{x^2}{1000}$$

d. Profit Function

Solution:

$$P(x) = R(x) - C(x)$$

e. Number of Units which maximizes the profit

Solution:

$$C^{'}(x) = R^{'}(x) \Rightarrow C^{'}(x) = 0.08x + 1 \Rightarrow R^{'}(x) = 100 - \frac{x}{500}$$

From this $x \cong 1238$

f. Find the rate of change of revenue in terms of rate of change of the units produced at a given production level $x = n$.

Solution:

$$\frac{dR}{dt} = R^{'}(x) * \frac{dx}{dt} = \left(100 - \frac{x}{500}\right) * \frac{dx}{dt}$$

$$\frac{dR}{dt}(n) = R^{'}(n) * \frac{dx}{dt} = \left(100 - \frac{n}{500}\right) * \frac{dx}{dt}$$

$$0 \le n \le 50000$$