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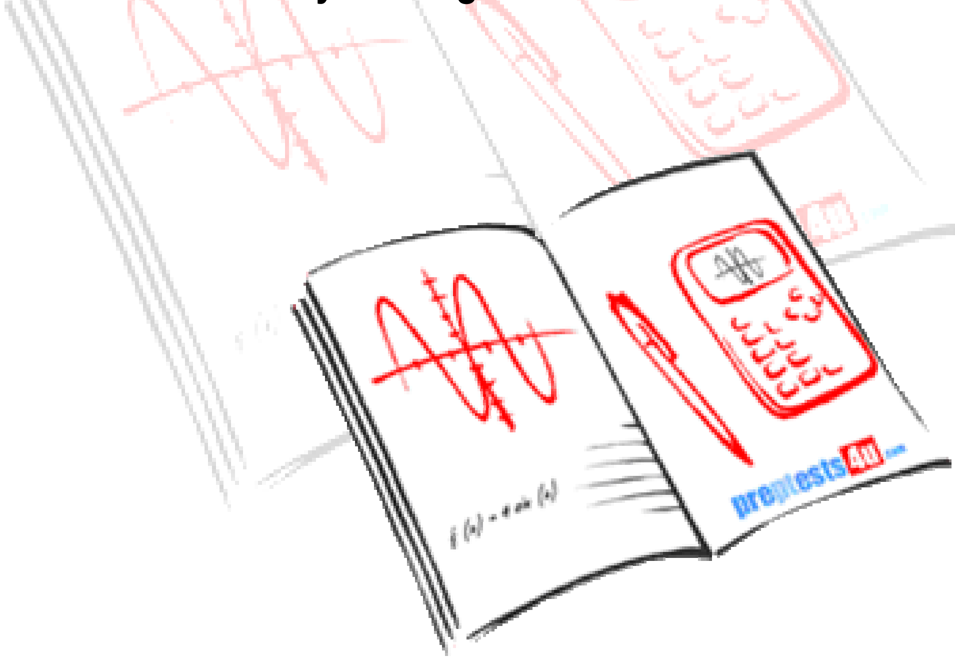
Calculus 1

Chapter 2 - The Concepts of Limit and Continuity

Conditions of Continuity for Single Variable Functions



Conditions of Continuity for Single Variable Functions

Definition of Continuity

Function $f(x)$ is continuous at a value a if the following conditions are satisfied.

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a} f(x) = f(a)$$

The above condition says that $\lim f(x)$ must exist as $x \to a$ (i.e., limits of the left and right must exist and equal to each other) and the value of the function $f(x)$ at a must be defined and equal to the limit of $f(x)$ as $x \to a$. Polynomials, Trigonometric Functions, and Rational Functions are among continuous functions in their respective domains.

The concept of continuity is further explored and clarified through following examples.

Example 1:

Let $f(x) = \dfrac{1 - x^2}{x^2 - 4x - 5}$

Find the followings:

a. $\lim_{x \to 5} f(x) = -\infty$, **using formula d., section 1 of this chapter.**

$\lim_{x \to 5^-} f(x) = -\infty$ and $\lim_{x \to 5^+} f(x) = -\infty$

b. $\lim_{x \to -1} f(x) = \dfrac{0}{0}$, as $x \to -1$, numerator and the denominator approach zero.

However, there are ways to get out of this situation, either by finding the common factor, causing both numerator and denominator to vanish, or using **l'Hospital's Rule,** as explained in Chapter 3. Fortunately, we have a choice of factoring to resolve this case.

$$\lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{1 - x^2}{x^2 - 4x - 5} = \lim_{x \to -1} \frac{(1 - x)(1 + x)}{(x - 5)(x + 1)} = \lim_{x \to -1} \frac{1 - x}{x - 5} = \frac{2}{-6} = -\frac{1}{3}$$

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) = -\frac{1}{3}$$

c. All the points of discontinuity and label them as removable or infinite (non-removable) discontinuity.

Since $f(x)$ is a rational function, the concern regarding the discontinuity is addressed to the denominator. To find the discontinuity (if any), we let the denominator to be equal zero.

$$x^2 - 4x - 5 = 0 \Rightarrow (x-5)(x+1) = 0 \Rightarrow x = 5 \,\&\, -1$$

$x = 5 \,\&\, x = -1$ are called points of discontinuity (singularities).

By plugging these values in the numerator, we find that $x = -1$ vanishes the numerator, hence $x = -1$ is a **removable discontinuity**. On the other hand, $x = 5$ is called an **infinite or non-removable discontinuity.**

Note:
Line $x = 5$ is called vertical asymptote and line $y = -1$ is called horizontal asymptote. To find the horizontal asymptote, find $\lim f(x)$ as $x \to \pm\infty$, as explained in the previous section.

Example 2:

Given the following function:

$$f(x) = \begin{cases} 2Sinx - 1 & , \quad x \le 0 \\ 1 - |-x| & , \quad 0 \prec x \le 1 \\ (x-1) & , \quad x \succ 1 \end{cases}$$

Find:

a. $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-}(2Sinx - 1) = -1$, since $x \to 0$ from left, we need to take limit of $2Sinx - 1$.

b. $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+}(1 - |-x|) = 1$, since $x \to 0$ from right, we need to take limit of $1 - |-x|$.

c. $f(0) = -1$ using equation $2Sinx - 1$ since $x = 0$ is in its domain.

<u>Conclusion</u>: $f(x)$ is not continuous at point $x = 0$ since Condition(s) of Continuity is <u>not</u> satisfied, $\lim f(x)$ as $x \to 0$, does not exist.

d. $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (1 - |-x|) = 0$, since $x \to 1$ from left, we need to take limit of $1 - |-x|$.

e. $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (x - 1) = 0$, since $x \to 1$ from right, we need to take limit of $(x - 1)$.

f $f(1) = 0$ since $x = 1$ belongs to the domain of $1 - |-x|$.

<u>Conclusion</u>: $f(x)$ is continuous at point $x = 1$ since all the conditions of continuity are satisfied.

Example 3:

For what value(s) of K the function $f(x)$ is continuous for all $x \in (-\infty, \infty)$.

$$f(x) = \begin{cases} e^x + 3 & , \quad x \geq 0 \\ x^2 + (k+1)x + k^2 & , \quad x \prec 0 \end{cases}$$

<u>Solution:</u>

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} (e^x + 3) = 4$ $\qquad x \to 0$ from right.

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} (x^2 + (k+1) + k^2) = k^2$ $\qquad x \to 0$ from left.

$f(0) = 4$

Condition of continuity at $x = 0$,

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = f(0) = 4$$

Hence,

$$k^2 = 4 \Rightarrow k = \pm 2$$

Note:
It is highly recommended to check back the answer(s) into the original equation(s) and verify the accuracy of the solutions.

The Intermediate Value Theorem

Assuming that function $f(x)$ is continuous on the interval $[a,b]$. Let p be any value between the values of $f(a)$ and $f(b)$. Hence, there is a point $t \in (a,b)$ such that $f(t) = p$.

Note:
If $f(a) \succ o$ and $f(b) \prec 0$
or $f(a) \prec o$ and $f(b) \succ 0$
then, based on this theorem, it is guaranteed that $f(x) = 0$ at $x = t_1 \in (a,b)$. t_1 is called a root of function $f(x)$ in the given interval.

Example 4:

Determine if the following functions have a **root** on the given interval, using **the Intermediate Value Theorem,** and give reason for your answer.

a. $f(x) = x^2 - 3x + 1$, $(1,3)$

$f(1) = -1$ and $f(3) = 1$

Solution:
Yes. $f(x)$ is continuous on the interval $[1,3]$ and the values of $f(x)$ at 1 and 3 have **opposite sign.**

b. $f(x) = x^2 - 3x + 1$, $(-1,3)$

$f(-1) = 5$ and $f(3) = 1$

Solution:
No. $f(x)$ is continuous on the interval $[-1,3]$ but the values of $f(x)$ at -1 and 3 have **same sign.**

Note:
Polynomials are continuous everywhere in their domains, $(-\infty,+\infty)$.

c. $f(x) = -3x + \frac{1}{2x} - 1$, $(-1,1)$

$f(-1) = \frac{3}{2}$ and $f(1) = -\frac{7}{2}$

Solution:
No. Although $f(x)$ is defined at -1 and $+1$, with opposite sign, this function is **not continuous at interval** $[-1,+1]$ **due to** $x = 0$**.**

d. $f(x) = 2Cosx + 2x - 3$, $(0,1)$

$f(0) = -1$, $f(1) \cong 0.0806$

Solution:
Yes. $f(x)$, sum of two continuous functions, is continuous on the interval $[0,1]$ and has opposite sign at the values 0 and 1.

Note:
You must set your calculator mode to radian before applying these values.