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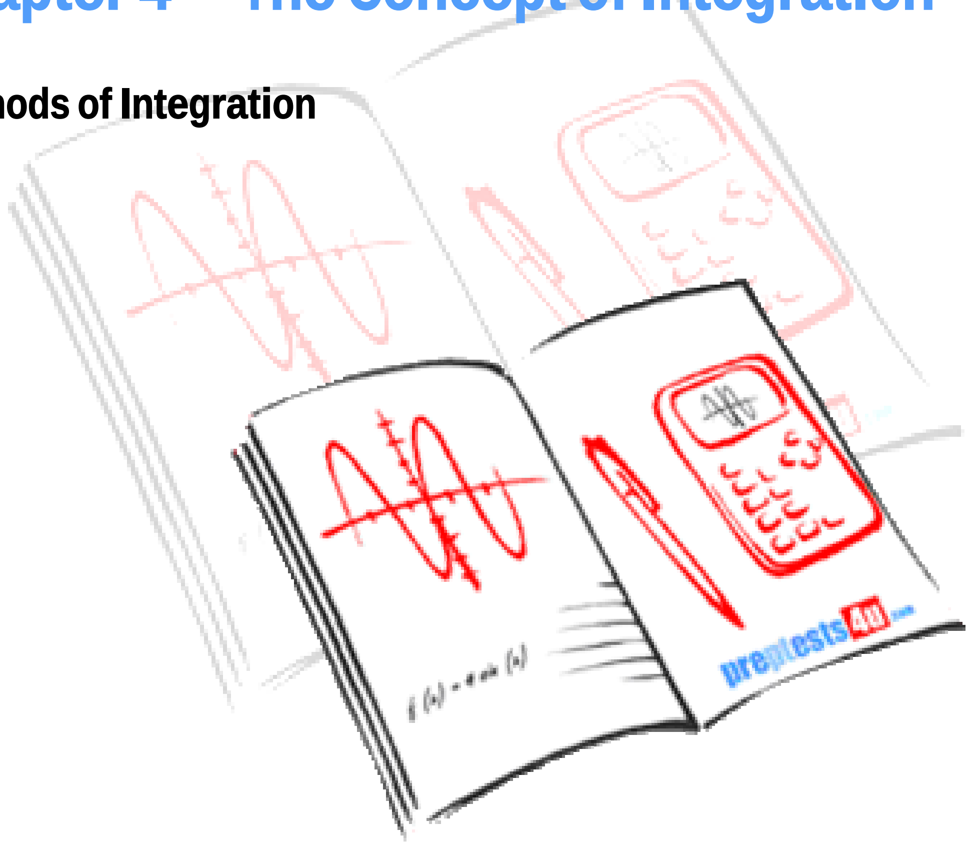
Business Calculus 1

Chapter 4 - The Concept of Integration

Methods of Integration



Methods of Integration

Substitution

Example 1:

$$\int \frac{x-2}{(x^2-4x)^3}dx$$

Let

$$u = x^2 - 4x\,,\; du = (2x-4)dx = 2(x-2)dx\,,\; \frac{1}{2}du = (x-2)dx$$

Hence,

$$\int \frac{x-2}{(x^2-4x)^3}dx = \int \frac{1}{2}\frac{du}{u^3} = -\frac{1}{4u^2} + C = -\frac{1}{4(x^2-4)^2} + C$$

Example 2:

$$\int \frac{x^2-2x+1}{(x^3-3x^2+3x)^3}dx$$

Let

$$u = x^3 - 3x^2 + 3x\,,\quad du = (3x^2-6x+3)dx = 3(x^2-2x+1)dx\,,$$
$$\frac{1}{3}du = (x^2-2x+1)dx$$

Hence,

$$\int \frac{x^2-2x+1}{(x^3-3x^2+3x)^3}dx = \int \frac{1}{3}\frac{du}{u^3} = -\frac{1}{6u^2} + C = -\frac{1}{6(x^3-3x^2+3x)^2} + C$$

Note:
To check the accuracy of the answer, take derivative of the right side which must equal $f(x)$.

Example 3:

$$\int_e^{e^2} \frac{1}{x}\left(Lnx^2\right)^2 dx$$

Let

$$u = \left(Lnx^2\right), \quad du = \frac{2}{x}dx, \quad \frac{1}{2}du = \frac{1}{x}dx$$

Hence,

$$\int_e^{e^2} \frac{1}{x}\left(Lnx^2\right)^2 dx = \int_2^4 \frac{1}{2}u^2 du = \left[\frac{u^3}{6}\right]_2^4 = \left[\frac{\left(Lnx^2\right)^3}{6}\right]_e^{e^2} = \frac{28}{3}$$

Example 4:

$$\int \frac{Lnx^3}{x}dx,$$

Let

$u = Lnx^3, \quad du = \frac{3}{x}dx, \quad \frac{1}{3}du = \frac{1}{x}dx$, hence

Hence,

$$\int \frac{Lnx^3}{x}dx = \int \frac{1}{3}udu = \frac{1}{6}u^2 + C = \frac{1}{6}\left(Lnx^3\right)^2 + C$$

Example 5:

$$\int \frac{\frac{1}{x}}{(Lnx)^3}dx$$

Let

$$u = Lnx, du = \frac{1}{x}dx$$

Hence,

$$\int \frac{\frac{1}{x}}{(Lnx)^3}dx = \int \frac{du}{u^3} = -\frac{1}{2u^2} + C = -\frac{1}{2(Lnx)^2} + C$$

Example 6:

$$\frac{d}{dx}\int_{2}^{x^2}\left(\frac{t-1}{t^2+3}\right)dt = \frac{x^2-1}{x^4+3}(2x)-\frac{1}{7}(0)=\frac{(2x)(x^2-1)}{x^4+3}=\frac{2x^3-2x}{x^4+3}$$

Note:
You may use formula 6, section 1, of this chapter.

Exercise:

Evaluate the following:

$$\frac{d}{dx}\int_{x-1}^{2x^2+1}\left(\frac{t-1}{t^2+3}\right)dt$$

Example 7:

$$\int_0^1 (x^2+2x+1)e^{(x^3+3x^2+3x)}dx$$

Let

$$u = x^3+3x^2+3x\,,\quad du = 3(x^2+2x+1)dx\;,\; u(0)=0, u(1)=7$$

Hence,

$$\int_0^1 (x^2+2x+1)e^{(x^3+3x^2+3x)}dx = \int_0^7 \frac{1}{3}e^u du = \left[\frac{e^u}{3}\right]_0^7 = \frac{1}{3}(e^7-1)$$

Example 8:

$$\int_0^{\frac{\pi}{2}} Sin(3x)e^{2Cos(3x)}dx$$

Let

$$u = 2Cos3x\,,\quad du = -6Sin(3x)dx\,,\; u(0)=2, u\left(\frac{\pi}{2}\right)=0$$

Hence,

$$\int_0^{\frac{\pi}{2}} Sin(3x)e^{2Cos(3x)}dx = \int_2^0 -\frac{1}{6}e^u du = -\left[\frac{e^u}{6}\right]_2^0 = -\frac{1-e^2}{6}$$

Example 9:

$$\int_0^{\frac{\pi}{4}} Sin(6x)e^{Cos^2(3x)}dx$$

Let

$$u = Cos^2(3x), \quad du = -6Sin(3x)Cos(3x)dx = -3Sin(6x)dx$$

$$u(0) = 1, u\left(\frac{\pi}{4}\right) = \frac{1}{2}$$

Hence,

$$\int_0^{\frac{\pi}{4}} Sin(6x)e^{Cos^2(3x)}dx = \int_1^{\frac{1}{2}} -\frac{1}{3}e^u du = \frac{e-\sqrt{e}}{3} \cong 0.3565$$

Integration by Parts

$$\int udv = uv - \int vdu$$

Example 10:

$$\int (x^2+1)Sin(4x)dx$$

Let

$$u = x^2+1, du = 2xdx \;, \quad dv = Sin(4x)dx, \;\; v = -\frac{1}{4}Cos(4x)$$

Hence,

$$\int (x^2+1)Sin(4x)dx = -\frac{1}{4}(x^2+1)Cos(4x) - \int -\frac{1}{2}xCos(4x)dx$$

Notice that the second integral on the right hand side requires to be evaluated using technique of integration by parts again. Hence, **as an exercise**, evaluate the following:

$$\int -\frac{1}{2}xCos(4x)dx$$

Note:
The choice of u and dv is not completely arbitrary. For the Example 10, do the following substitutions. **You will end up with a huge headache and/or possibly dizziness.**

Let

$$u = Sin(4x) \quad \text{and} \quad dv = (x^2+1)dx$$

$$v = \frac{1}{3}x^3 + x$$

Have you ever tried to back up in the highway?!
Please do not do that. It is very dangerous.
Mucho Gracias!

Integration by Partial Fraction (Rational Fraction)

Based on the previously defined rational fractions, $f(x) = \frac{R(x)}{S(x)}$, we outline steps in using partial fraction method to integral problems.

a. If the highest power of the numerator is equal or higher than the highest power of the denominator, then use the division rule to obtain a rational fraction which has power of numerator lower than that of the denominator. You may use the partial fraction technique, if applicable, to solve the problem.

b. If the highest power of the numerator is less than the highest power of the denominator, then apply the method of partial fraction. However, it is a good idea to investigate the fraction before applying lengthy method of partial fraction as illustrated in the following examples. First try if the method of substitution works, otherwise attempt the partial fraction technique.

Example 11:

$$\int \frac{x^4}{x^3+1}dx$$

Solution:

$$\int \frac{x^4}{x^3+1}dx = \int \frac{x^4+x-x}{x^3+1}dx = \int \frac{x(x^3+1)-x}{x^3+1}dx = \int xdx - \int \frac{x}{x^3+1}dx$$

$$\int xdx = \frac{x^2}{2} + c_1$$

$$\int \frac{x}{x^3+1}dx = \int \frac{x}{(x+1)(x^2-x+1)}dx = \int \frac{A}{x+1}dx + \int \frac{Bx+C}{x^2-x+1}dx$$

Partial fraction method:

$$\frac{x}{(x+1)(x^2-x+1)} = \frac{A}{(x+1)} + \frac{Bx+C}{x^2-x+1}$$

After taking common denominator of the right hand side and simplifying,

$$(A+B)x^2 + (B+C-A)x + (A+C) = x$$

From this:

$$A+B=0, A+C=0, B+C-A=1$$

Solving three equations and three unknowns, we get

$$A = -\frac{1}{3}, B = C = \frac{1}{3}$$

$$\int \frac{x}{x^3+1}dx = \int \frac{x}{(x+1)(x^2-x+1)}dx = \int \frac{-\frac{1}{3}}{x+1}dx + \int \frac{\frac{1}{3}x+\frac{1}{3}}{x^2-x+1}dx$$

$$\int \frac{-\frac{1}{3}}{x+1}dx = -\frac{1}{3}Ln|x+1| + c_2$$

$$\int \frac{\frac{1}{3}x+\frac{1}{3}}{x^2-x+1}dx = \frac{1}{6}\int \frac{2x-1}{x^2-x+1}dx + \int \frac{\frac{1}{2}}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}dx = \frac{1}{6}Ln\left(x^2-x+1\right) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + c_3$$

Note:
The problem was broken into few simpler integrals and evaluated separately. Carrying this mess at one shot would be like holding two watermelons in one hand! Imagine the result!

Example 12:

$$\int \frac{(2-x)^2}{x}dx$$

Solution:

$$\int \frac{(2-x)^2}{x}dx = \int \frac{x^2-4x+4}{x}dx = \int xdx - 4\int dx + 4\int \frac{1}{x}dx$$

$$\int \frac{(2-x)^2}{x}dx = \frac{x^2}{2} - 4x + 4Ln|x| + C$$

This is an example of a trivial one which denominator is a single term and one may apply the formulas of integration directly.

Exercise:

Evaluate the following integral

$$\int \frac{x^2}{(2-x)^2}dx$$