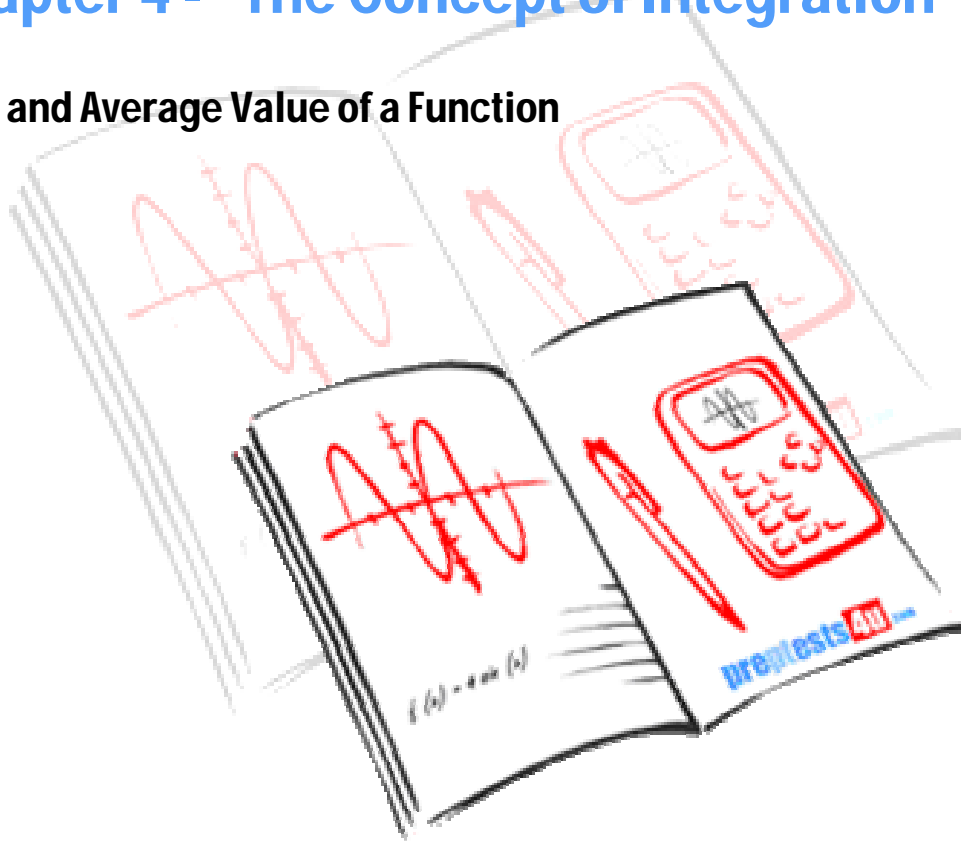


# Business Calculus 1

## Chapter 4 - The Concept of Integration

### Area and Average Value of a Function



## Area and Average Value of a Function

The following formula evaluates the area between two curves  $f(x)$  and  $g(x)$  bounded by lines  $x = a$  and  $x = b$ .

$$A = \int_a^b |f(x) - g(x)| dx$$

The absolute value takes care of both conditions  $f(x) \geq g(x)$  or  $g(x) \geq f(x)$  on  $[a, b]$ . If one of the functions  $f(x)$  or  $g(x)$  is equal to zero, then

$$A = \int_a^b |f(x)| dx \text{ or } A = \int_a^b |g(x)| dx \text{ will be the area under a curve on } [a, b].$$

### Example 1:

Find the area under the curve  $f(x) = 3x - 2$ ,  $x = -2$  and  $x = 4$ .

**Solution:**

$$A = \int_{-2}^4 (3x - 2) dx = \left[ \frac{3x^2}{2} - 2x \right]_{-2}^4 = 6$$

### Example 2:

Find the area under the curve  $f(x) = |3x - 2|$ ,  $x = -2$  and  $x = 4$ .

**Solution:**

Since the absolute value is given here, we must find out the signs of  $f(x)$  on the given interval.

$$3x - 2 = 0, x = \frac{2}{3}$$

$$f(x) < 0 \text{ on } \left(-2, \frac{2}{3}\right) \text{ and } f(x) \geq 0 \text{ on } \left[\frac{2}{3}, 4\right], \text{ hence}$$

$$A = \int_{-2}^4 f(x)dx = -\int_{-2}^{\frac{2}{3}} (3x-2)dx + \int_{\frac{2}{3}}^4 (3x-2)dx$$

To evaluate these integrals, follow the solutions in Example 1.

**Example 3:**

Find the area under the curve  $f(x) = x^2 - 3x + 2$ ,  $x = -1$  and  $x = 2$ .

**Solution:**

$$A = \int_{-1}^2 (x^2 - 3x + 2)dx = \left[ \frac{x^3}{3} - \frac{3x^2}{2} + 2x \right]_{-1}^2 = \frac{9}{2}$$

**Example 4:**

Find the area under the curve  $f(x) = |x^2 - 3x + 2|$ ,  $x = -1$  and  $x = 3$ .

**Solution:**

Since the absolute value is given here, we must find out the sign of  $f(x)$  on the given interval.

$$x^2 - 3x + 2 = (x-1)(x-2) = 0, x = 1, x = 2$$

$$f(x) < 0 \text{ on } (1,2) \text{ and } f(x) \geq 0 \text{ on } [-1,1] \cup [2,3]$$

Hence,

$$A = \int_{-1}^3 f(x)dx = \int_{-1}^1 (x^2 - 3x + 2)dx - \int_1^2 (x^2 - 3x + 2)dx + \int_2^3 (x^2 - 3x + 2)dx$$

To evaluate these integrals, follow the solutions in Example 1.

**Example 5:**

Find the area bounded by  $f(x) = \frac{e^x + 1}{x + e^x}$ ,  $x = 0$ ,  $x = \text{Ln}3$ .

**Solution:**

$$A = \int_0^{\text{Ln}3} f(x)dx = \int_0^{\text{Ln}3} \frac{e^x + 1}{x + e^x} dx$$

Let

$$u = x + e^x, \quad du = (1 + e^x)dx, \quad u(0) = 1, u(\text{Ln}3) = 3 + \text{Ln}3$$

Hence,

$$A = \int_0^{\text{Ln}3} f(x)dx = \int_0^{\text{Ln}3} \frac{e^x + 1}{x + e^x} dx = \int_1^{3+\text{Ln}3} \frac{du}{u} = [\text{Ln}u]_1^{3+\text{Ln}3} = \text{Ln}(3 + \text{Ln}3) \cong 1.4106$$

**Exercise:**

Find the area bounded by  $f(x) = \frac{e^{-x} - 1}{x + e^{-x}}$ ,  $x = 0$ ,  $x = \text{Ln}3$ .

**Average Value of a Function**

Average value of a function is defined as:

$$f_{ave} = \frac{1}{b-a} \int_a^b f(x)dx$$

**Example 6:**

Find the average value of  $f(x)$  on  $[-2, 3]$

$$f(x) = \begin{cases} e^x + 3, & x \geq 0 \\ x^2 + 3x + 4, & x < 0 \end{cases}$$

**Solution:**

$$f_{ave} = \frac{1}{2} \int_{-2}^0 (x^2 + 3x + 4)dx + \frac{1}{3} \int_0^3 (e^x + 3)dx$$

Evaluations of these integrals are very straight forward. However, it is absolutely important to identify the signs of  $f(x)$  on the given interval before any attempt to do the integration.